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W_q^l-REGULARITY OF SOLUTIONS TO SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

We obtain a solution, in a sense final from the standpoint of the theory of Sobolev spaces, to the problem of W_q^l-regularity of solutions to a system of (generally) nonlinear partial differential equations in the case where the system is locally close to elliptic systems of linear equations with constant coefficients (see Theorem 9).

This result gives a number of important consequences for solutions as to nonlinear systems as to linear systems.

The main corollary to Theorem 9, in the case of nonlinear systems, is Theorem 1 claiming that the higher derivatives of an elliptic C^l-smooth solution f to a system of lth-order nonlinear partial differential equations constructed from C^1-smooth functions meet a Hölder condition with every exponent α, $0 < \alpha < 1$, locally in $\operatorname{dom} f$ (the assertion is an extension to nonlinear systems of arbitrary order of Nirenberg's and Morrey's results on Hölder continuity of the higher derivatives of solutions to a single second-order elliptic equation and to a second-order elliptic system respectively).

In the case of linear systems, Theorem 9 implies Theorem 2 by which if a system of linear partial differential equations of order l with measurable coefficients and right-hand sides is uniformly elliptic then, under the hypothesis of a (sufficiently) slow variation of its leading coefficients, the degree of local integrability of lth-order partial derivatives of every $W^l_{q,\mathrm{loc}}$-solution, $q > 1$, to the system coincides with the degree of local integrability of the lower coefficients and the right-hand sides. Moreover, Theorem 2 is naturally complemented by Theorem 3 and 4 which imply that if $q_0 \to \infty$ then the degree of vanishing of the quantity $\mathcal{E}_{t,r}(q_0)$ representing the least upper bound of the set of numbers $\varepsilon \geq 0$ such that every $W^l_{r,\mathrm{loc}}$-solution to any uniformly elliptic linear system with measurable coefficients and right-hand sides the parameter of local variation of the leading coefficients of which is at most ε and the degree q_0 of local integrability of the remaining coefficients and the right-hand sides of which is greater than n is the same as the degree of vanishing of the function $q_0 \mapsto 1/q_0$.

Here we expose some results obtained in the recent 5–6 years. These results concern the problem of W_q^l-regularity of solutions to systems of (generally) nonlinear partial differential equations. The systems involved are locally close to the elliptic systems of linear equations with constant coefficients.

We begin with a discussion of the two major consequences of the main result of this article, Theorem 9 (we will consider Theorem 9 itself thereafter). The first consequence (Theorem 1) applies to nonlinear systems and second (Theorem 2) covers the case of linear systems.

§1. Nonlinear systems.

Consider the system

$$\begin{gathered} L_j(x; f_1(x), \dots, f_m(x); \dots, \partial^{p_1} f_\varkappa(x), \dots; \dots, \partial^{p_2} f_\varkappa(x), \dots \\ \dots; \dots; \dots, \partial^{p_l} f_\varkappa(x), \dots) = 0, \quad j = 1, 2, \dots, k, \end{gathered} \tag{1}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ ($\mathbb{R}^n$ is the n-dimensional real arithmetic Euclidean space), $p_\nu = (p_{\nu 1}, p_{\nu 2}, \dots, p_{\nu n})$ is a multi-index of order $|p_\nu| = \sum_{s=1}^{n} p_{\nu s} = \nu = 0, 1, 2, \dots, l$, $\partial^{p_\nu} f_\varkappa = [(\partial_1)^{p_{\nu 1}} \circ (\partial_2)^{p_{\nu 2}} \circ \dots \circ (\partial_n)^{p_{\nu n}}] f_\varkappa$ ($\partial_s = \partial / \partial x_s$) is the partial derivative of the function $f_\varkappa$

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($= \partial^{p_0} f_\varkappa$) corresponding to p_ν, and (1) includes the symbols of all such partial derivatives of each function $f_\varkappa$, $\varkappa = 1, 2, \dots, m$, up to lth order. This is an lth-order system of k nonlinear partial differential equations in m sought real functions $f_\varkappa$ of n real variables $x_1, x_2, \dots, x_n$ ($n \geq 2$, $m \geq 1$, $k \geq 1$, $l \geq 1$). Here $\mathfrak{L}_j$ are continuous real functions having continuous first-order partial derivatives with respect to all their arguments (i. e., functions of class C^1):

$$\mathfrak{L}_j = \mathfrak{L}_j(x; \dots, v_{p_0,\varkappa}, \dots; \dots, v_{p_1,\varkappa}, \dots; \dots; \dots, v_{p_l,\varkappa}, \dots), \tag{2}$$

where $(x; \dots, v_{p_0,\varkappa}, \dots; \dots, v_{p_1,\varkappa}, \dots; \dots; \dots, v_{p_l,\varkappa}, \dots) = y \in Y$, Y is an open set in $\mathbb{R}^{N_l}$ ($Y \Subset \mathbb{R}^{N_l}$), $N_l = n + m \sum\limits_{\nu=0}^{l} n_\nu$ with $n_\nu = \frac{(n+\nu-1)!}{\nu!(n-1)!}$ ($\nu = 0, 1, 2. \dots, l$).

DEFINITION 1. *System (1) is called elliptic if*

$$\operatorname{rank}\left\{\sum_{p_l} \zeta^{p_l} \begin{pmatrix} \partial_{v_{p_l,1}}\mathfrak{L}_1(y) & \dots & \partial_{v_{p_l,m}}\mathfrak{L}_1(y) \\ \dots & \dots & \dots \\ \partial_{v_{p_l,1}}\mathfrak{L}_k(y) & \dots & \partial_{v_{p_l,m}}\mathfrak{L}_k(y) \end{pmatrix}\right\} = m \tag{3}$$

for all $\zeta \in \mathbb{R}^n \setminus \{0\}$ and $y \in Y$ (the sum in (3) is taken over all multi-indices p_l of order $|p_l| = l$). A solution $f : U \to \mathbb{R}^m$ ($U \Subset \mathbb{R}^n$) of class $C^l(U, \mathbb{R}^m)$ to system (1) is called elliptic if ellipticity condition (3) is fulfilled at each $y = (x; \dots, \partial^{p_0} f_\varkappa(x), \dots; \dots \dots, \partial^{p_1} f_\varkappa(x), \dots; \dots; \dots, \partial^{p_l} f_\varkappa(x), \dots)$, $x \in U$.

REMARK *It is well known that there are "exceptional" collections of n, m, k, and l for which systems (1) do not have elliptic solutions. For example, every set of four numbers n, m, k, and l such that $m = k = 1$ and l is odd represents an exceptional collection in this sense. Below, we assume that the quadruples n, m, k, and l are not exceptional.*

The following assertion holds (see [1], [2]):

THEOREM 1. *Suppose that the functions $\mathfrak{L}_j$ in* (1), (2) *belong to $C^1(Y)$. Then the lth-order partial derivatives of every elliptic C^l-solution $f : U \to \mathbb{R}^m$, $U \Subset \mathbb{R}^n$, to* (1) *satisfy a Hölder condition with an arbitrary exponent α in $]0, 1[$ locally in U: if $0 < \alpha < 1$ and E is a compact subset of U then there exists a number $C_{\alpha,E} \geq 0$ such that*

$$|\partial^{p_l} f_\varkappa(x') - \partial^{p_l} f_\varkappa(x'')| \leq C_{\alpha,E}|x' - x''|^\alpha,$$

$x', x'' \in E$, $|p_l| = l$, $\varkappa = 1, 2, \dots, m$.

Theorem 1 is an extension of the well-known results by Nirenberg and Morrey on the Hölder continuity of higher derivatives of solutions to a single elliptic second-order equation [3] and to an elliptic second-order system [4].

§2. Linear systems.

Suppose that an lth-order system

$$\sum_{|p| \leq l} a_p(x) \partial^p f(x) = h(x), \quad x \in U, \tag{4}$$

of k linear partial differential equations in m sought real functions of n real variables $x_1, x_2, \dots, x_n$, $n \geq 2$, $m \geq 1$, $k \geq 1$, $l \geq 1$, has measurable coefficients $a_p^{j\varkappa}$ and right-hand sides h_j, $j = 1, 2, \dots, k$, $\varkappa = 1, 2, \dots, m$, $|p| \leq l$ (in (4) $U \Subset \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$,

$f = (f_1, f_2, \ldots, f_m) : U \to \mathbb{R}^m$, $h = (h_1, h_2, \ldots, h_k) : U \to \mathbb{R}^k$, and $a_p(x) = \left(a_p^{j\varkappa}(x)\right)_{\substack{j=1,2,\ldots,k\\ \varkappa=1,2,\ldots,m}}$ is a real $(k \times m)$-matrix). Moreover, assume that the following conditions are fulfilled:

($\circ$) System (4) is *uniformly elliptic*: there exists a number $t \in]0, \infty[$ such that

$$|a_p^{j\varkappa}(x)| \le t, \quad |p| = l, \quad j = 1, 2, \ldots, k, \quad \varkappa = 1, 2, \ldots, m,$$

and

$$\left|\sum_{|p|=l} \zeta^p a_p(x) u\right| \ge \frac{1}{t}, \quad \zeta \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad |\zeta| = 1, \quad |u| = 1,$$

for almost every $x \in U$.

($\circ\circ$) The leading coefficients $a_p^{j\varkappa}$, $|p| = l$, possess *the property of slow variation*, i. e., there exists a nonnegative number ε such that every point $x \in U$ has a neighborhood U_x ($\subset U$) and a measurable set $E \subset U_x$, $\operatorname{mes}(U_x \setminus E) = 0$, for which the inequalities

$$|a_p^{j\varkappa}(x') - a_p^{j\varkappa}(x'')| \le \varepsilon, \quad |p| = l, \quad j = 1, \ldots, k, \quad \varkappa = 1, \ldots, m,$$

hold for every two points x' and x'' in E.

($\circ\circ\circ$) There exists a number $q_0 \in [1, \infty[$ such that the remaining coefficients $a_p^{j\varkappa}$, $|p| < l$, and the right-hand sides h_j belong to $L_{q_0,\mathrm{loc}}(U, \mathbb{R})$.

Remark 1. *Conditions* ($\circ$) *and* ($\circ\circ$) *are not independent: condition* ($\circ$) *implies condition* ($\circ\circ$) *with* $\varepsilon = 2t$. *It is important that the parameter* ε *of slow variation of leading coefficients of (4) can take any value less than* $2t$.

Remark 2. *A typical example of systems (4) with* ($\circ$) – ($\circ\circ\circ$) *is given by a Beltrami system*

$$\partial_{\bar z} f(z) = q(z) \partial_z f(z),$$

$$\|q\|_\infty = \operatorname*{ess\,sup}_{z \in \operatorname{dom} q} |q(z)| \le \varepsilon < 1$$

(here we use the standard complex notations: $f : U \to \mathbb{C}$, $q : U \to \mathbb{C}$, U *is an open subset in the field* $\mathbb{C}$ *of complex numbers,* $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$, $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, i *is the imaginary unit). And what is more, its various multidimensional generalizations, as well as linear uniformly elliptic systems with continuous coefficients and right-hand sides, are among systems like (4) satisfying* ($\circ$) – ($\circ \circ \circ$).

Definition 2. *A solution of class* $W_{q,\mathrm{loc}}^l(U, \mathbb{R}^m)$ *(*$W_{q,\mathrm{loc}}^l$*-solution),* $q \ge 1$, *to system (4) (and, also, to all systems below) is a mapping* $f \in W_{q,\mathrm{loc}}^l(U, \mathbb{R}^m)$ *meeting (4) almost everywhere in* U.

Remark 3. $W_q^l(U, \mathbb{R}^m)$ *is the Sobolev space of all mappings* $g = (g_1, g_2, \ldots, g_m) : U \to \mathbb{R}^m$ *whose every component function* $g_\varkappa$, $\varkappa = 1, 2, \ldots, m$, *belongs to the Lebesgue space* $L_q(U)$ *and has all weak partial derivatives up to* l*th order integrable to the power* q *in* U; $W_{q,\mathrm{loc}}^l(U, \mathbb{R}^m)$ *is the space of all mappings* $g : U \to \mathbb{R}^m$ *with the property that each point* $x \in U$ *has a neighborhood* $U_x \subset U$ *such that* $g|_{U_x} \in W_q^l(U_x, \mathbb{R}^m)$.

Furthermore, let $\mathcal{O} = \mathcal{O}^{n,m,k,l}$ ($n \ge 2$, $m \ge 1$, $k \ge 1$, $l \ge 1$) be the set of all lth-order elliptic linear differential operators with constant coefficients of the form

$$D = (D_1, \ldots, D_k) = \sum_{|p|=l} a_p \partial^p, \tag{5}$$

where $a_p = \left(a_p^{j\varkappa}\right)_{\substack{j=1,\dots,k\\ \varkappa=1,\dots,m}}$ is a real $(k \times m)$-matrix. Recall [5] that operator (5) is elliptic if (and only if) its symbol $\sigma_D(\zeta) = \sum\limits_{|p|=l} \zeta^p a_p = \sum\limits_{|p|=l} (\zeta_1)^{p_1} \dots (\zeta_n)^{p_n} a_p$ satisfies the condition rank $\sigma_D(\zeta) = m$ for all $\zeta \in \mathbb{R}^n \setminus \{0\}$. It should be noted that the ellipticity of operator (5) is equivalent to the ellipticity of the system $Dg = 0$ in the sense of Definition 1.

Finally, define the following set $\mathcal{O}_t = \mathcal{O}_t^{n,m,k,l}$ of elliptic linear differential operators with constant coefficients:

$$\mathcal{O}_t = \Big\{ D = \sum_{|p|=l} a_p \partial^p \in \mathcal{O}, \quad \left|a_p^{j\varkappa}\right| \le t, \quad j = 1, \dots, k, \quad \varkappa = 1, \dots, m, \quad |p| = l,$$

$$\inf_{\zeta \in \mathbb{R}^n,\, v \in \mathbb{R}^m,\, |\zeta|=1,\, |v|=1} \Big| \sum_{|p|=l} \zeta^p a_p v \Big| \ge 1/t \Big\}. \tag{6}$$

We also denote by $t_0 = t_0^{n,m,k,l}$ the least of t such that $\mathcal{O}_t \neq \emptyset$.

Note that, immediately from (6), we obtain

LEMMA 1. *The relations $t_0 \ge \frac{1}{k^{1/4}}$, $\mathcal{O}_{t'} \subset \mathcal{O}_{t''}$ if $t' < t''$, and $\bigcup\limits_{t \ge t_0} \mathcal{O}_t = \mathcal{O}$ hold.*

Granted Lemma 1 and the relations $\mathcal{O}_t = \emptyset$, $0 < t < t_0$, from now on, we assume that $t_0 \le t < \infty$.

The following assertion (see [6]) is main in this section:

THEOREM 2. *Let numbers t, q_0, and r satisfy the inequalities $t_0 \le t < \infty$, $n < q_0 < \infty$, and $1 < r < \infty$ respectively. Then there exists a positive number $\varepsilon_{t,q_0,r} = \varepsilon_{t,q_0,r}^{n,m,k,l}$ having the property: if system (4) with measurable coefficients and right-hand sides satisfies the conditions $(\circ) - (\circ\circ\circ)$ with these t and q_0, and with $\varepsilon < \varepsilon_{t,q_0,r}$, then every $W^l_{r,\mathrm{loc}}$-solution to the system is its $W^l_{q_0,\mathrm{loc}}$-solution.*

In other words, Theorem 2 claims that if system (4) with measurable coefficients and right-hand sides is uniformly elliptic then, under the hypothesis of sufficiently slow variation of its leading coefficients $a_p^{j\varkappa}$, $|p| = l$, the degree of local integrability of lth-order partial derivatives of each $W^l_{r,\mathrm{loc}}$-solution to the system is the same as the degree of local integrability of its lower coefficients and right-hand sides.

Starting from Theorem 2 and fixing t, r, and q_0 such that $t_0 \le t < \infty$, $1 < r < q_0$, and $n < q_0 < \infty$, we introduce the quantity $\mathcal{E}_{t,r}(q_0) = \mathcal{E}_{t,r}^{n,m,k,l}(q_0)$ representing the least upper bound of the set of numbers $\varepsilon \ge 0$ such that every $W^l_{r,\mathrm{loc}}$-solution to any system (4) with measurable coefficients and right-hand sides meeting $(\circ) - (\circ\circ\circ)$ with the values of ε, t, and q_0 considered now, belongs to $W^l_{q_0,\mathrm{loc}}$.

Using $\mathcal{E}_{t,r}(q_0)$, we can strengthen Theorem 2 as follows:

THEOREM 3. *For each $r \in \,]1, \infty[$, there exists a function $C_r(t) = C_r^{n,m,k,l}(t)$, $0 < C_r(t) < \infty$, of t such that*

$$\frac{C_r(t)}{q_0} \le \mathcal{E}_{t,r}(q_0)$$

if $n < q_0 < \infty$.

THEOREM 4. *Let $(m-1)^2 + (l-1)^2 \neq 0$. There exists a number $\bar{q}_0(t) = \bar{q}_0^{n,m,k,l}(t) > n$, depending on n, m, k, l, and t, such that the inequality*

$$\mathcal{E}_{t,r}(q_0) \le \frac{C}{q_0},$$

where the quantity $C = C(n, m, k, l)$, $0 < C < \infty$, depends only on n, m, k, and l, holds for each pair of numbers q_0 and r, $\bar{q}_0(t) < q_0 < \infty$, $1 < r < q_0$.

COROLLARY. *Let $(m-1)^2 + (l-1)^2 \neq 0$. If t is fixed then $\mathcal{E}_{t,r}(q_0) \to 0$ as $q_0 \to \infty$ uniformly over r. Furthermore, if the values of t and r are fixed simultaneously then the degree of vanishing of $\mathcal{E}_{t,r}(q_0)$ as $q_0 \to \infty$ is the same as the degree of vanishing of the function $q_0 \mapsto 1/q_0$*

Theorem 2, the definition of $\mathcal{E}_{t,r}(q_0)$ and the embedding theorems for the Sobolev spaces in turn imply

THEOREM 5. *Let $1 < r < \infty$. Suppose that system (4) has measurable coefficients and right-hand sides and satisfies the conditions $(\circ) - (\circ\circ\circ)$ where $t_0 \leq t < \infty$, $n < q_0 < \infty$, and $\varepsilon < \mathcal{E}_{t,r}(q_0)$. Then each $W^l_{r,\mathrm{loc}}$-solution $f : U \to \mathbb{R}^m$ to (4) belongs to $C^{(l-1)+(1-n/q_0)}_{\mathrm{loc}}(U, \mathbb{R}^m)$.*

Note that by $C^{\mu+\alpha}_{\mathrm{loc}}(U, \mathbb{R}^m)$, $U \subset \mathbb{R}^n$, we mean the space of all continuous mappings $g : U \to \mathbb{R}^m$ having all partial derivatives up to order μ continuous in U, with the μth-order derivatives meeting the Hölder condition with exponent α $(0 < \alpha < 1)$ locally in U.

Theorem 5 is naturally complemented by Theorem 6 claiming that, in the case when system (4) is first-order (i. e., $l = 1$), the condition $q_0 > n$ in Theorem 5 is exact.

THEOREM 6. *If $l = 1$ and $1 \leq q_0 \leq n$ then, for every pair of numbers t and ε such that $t_0 \leq t < \infty$ and $0 \leq \varepsilon < \infty$, there exists a system of the kind of (4) that meets conditions $(\circ) - (\circ\circ\circ)$ with the indicated values of ε, q_0, and t and has an unremovably discontinuous $W^1_{q_0,\mathrm{loc}}$-solution.*

In conclusion of the section, we remark that Theorems 2, 5 and 6 imply (together with Lemma 4 of Section 3 and the Remark to it) the following two assertions (see [6]):

THEOREM 7. *If the coefficients $a_p^{j\varkappa}$ and right-hand sides h_j $(j = 1, 2, \ldots, k;\ \varkappa = 1, 2, \ldots, m;\ |p| \leq l)$ of system (4) of linear partial differential equations are continuous functions and the system itself is elliptic (i. e., $\operatorname{rank}\big\{\sum_{|p|=l} \zeta^p a_p(x)\big\} = m$ for every pair of points $\zeta \in \mathbb{R}^n \setminus \{0\}$ and $x \in U$) then, for arbitrary $r \in\,]1, \infty[$, every $W^l_{r,\mathrm{loc}}$-solution to the system is its $W^l_{q,\mathrm{loc}}$-solution for every $q \in [1, \infty[$ and, consequently, belongs to each space $C^{(l-1)+\alpha}_{\mathrm{loc}}(U, \mathbb{R}^m)$, $0 < \alpha < 1$.*

THEOREM 8. *Suppose that the higher coefficients $a_p^{j\varkappa}$, $|p| = l$ $(j = 1, 2, \ldots, k;\ \varkappa = 1, 2, \ldots, m)$, of (4) are continuous functions. Assume that there exists a number $q_0 \geq 1$ such that the remaining coefficients $a_p^{j\varkappa}$, $|p| < l$, and the right-hand sides h_j belong to $L_{q_0,\mathrm{loc}}(U, \mathbb{R})$. Finally, assume that the system is elliptic (this means the same as in Theorem 7). Then, in the case $q_0 > n$ and $1 < r < \infty$, every $W^l_{r,\mathrm{loc}}$-solution to the system is its $W^l_{q_0,\mathrm{loc}}$-solution and, hence, belongs to $C^{(l-1)+(1-n/q_0)}_{\mathrm{loc}}(U, \mathbb{R}^m)$, and if $1 \leq q_0 \leq n$ then there exists a first-order system (4) of the kind under consideration which has an unremovably discontinuous $W^1_{q_0,\mathrm{loc}}$-solution.*

§3. The general case.

Consider a system of lth-order (in general) nonlinear partial differential equations

$$\begin{gathered}\mathfrak{L}_j(x; v_{l-1}(f,x); v^l(f,x)) = \\ \mathfrak{L}_j(x; v^0(f,x); v^1(f,x); \ldots; v^{l-1}(f,x); v^l(f,x)) = 0, \quad j = 1, 2, \ldots, k,\end{gathered} \tag{7}$$

where $x = (x_1, x_2, \dots, x_n)$; $f = (f_1, f_2, \dots, f_m)$; $v^\nu(f,x) = (\dots, \partial^{p_\nu} f_\varkappa(x), \dots)$ is the collection of the values of all partial derivatives $\partial^{p_\nu} f_\varkappa(x)$, p_ν is a multi-index of order ν, of all functions $f_\varkappa$, $\varkappa = 1, 2, \dots, m$, at x, ordered, say, lexicographically; $v_{l-1}(f,x) = (v^0(f,x); v^1(f,x); \dots; v^{l-1}(f,x))$; the functions $\mathfrak{L}_j$ are real and defined on the set $U_1 = U \times \prod_{\nu=0}^{l} (\mathbb{R}^m)^{n_\nu} \subset \mathbb{R}^{N_l}$, $U \Subset \mathbb{R}^n$, $N_l = n + m \sum_{\nu=0}^{l} n_\nu$, $n_\nu = \frac{(n+\nu-1)!}{\nu!(n-1)!}$, $\nu = 0, 1, \dots, l$. System (7) consists of k equations in m sought real functions $f_\varkappa$, $\varkappa = 1, 2, \dots, m$, of n real variables $x_1, x_2, \dots, x_n$, and satisfies the following. For almost every $x \in U$, the functions $\mathfrak{L}_j$, $j = 1, 2, \dots, k$, take finite values $\mathfrak{L}_j(x; v_{l-1}; v^l)$ whenever $(v_{l-1}; v^l) = (v^0; v^1; \dots; v^{l-1}; v^l) \in \prod_{\nu=0}^{l} (\mathbb{R}^m)^{n_\nu}$, $v^\nu = (\dots, v_{p_\nu,\varkappa}, \dots) \in (\mathbb{R}^m)^{n_\nu}$; moreover, $\mathfrak{L}$ satisfies the following conditions:

(i) *The functions* $V_j(x; v_{l-1}; v^l) = \mathfrak{L}_j(x; v_{l-1}; v^l) - \mathfrak{L}_j(x; v_{l-1}; 0)$, $j = 1, 2, \dots, k$, *are measurable and*

$$|V(x; v_{l-1}; v^l)| \le \eta(x)|v^l|$$

if $(v_{l-1}; v^l) \in \mathbb{R}^{N_l - n}$ *for almost every* $x \in U$. *Here* η *is a nonnegative measurable function on* U *locally bounded in the* sup-norm $\|\cdot\|_\infty$: every point $x \in U$ has a neighborhood Z such that $\|\eta|_Z\|_\infty = \operatorname{ess\,sup}_{z \in Z} \eta(z) < \infty$.

(ii) *The mapping* $(x; v_{l-1}) \mapsto T(x; v_{l-1}) = \mathfrak{L}(x; v_{l-1}; 0)$, $(x; v_{l-1}) \in U \times \mathbb{R}^{N_{l-1} - n}$, *is measurable. Furthermore, there exists* $q_0 > n$ ($q_0 < \infty$) *such that*

a) *for almost every* $x \in U$, *the mapping* $T(x; \cdot) : \mathbb{R}^{N_{l-1} - n} \to \mathbb{R}^k$ *meets the Lipschitz condition*

$$|T(x; v'_{l-1}) - T(x; v''_{l-1})| \le E(x)|v'_{l-1} - v''_{l-1}|, \quad v'_{l-1}, v''_{l-1} \in \mathbb{R}^{N_{l-1} - n},$$

where E *is a function locally integrable to the power* q_0 *in* U ($E \in L_{q_0,\mathrm{loc}}(U, \mathbb{R})$);

b) *the mapping* $T(\cdot; 0) : x \mapsto T(x; 0)$, $x \in U$, *belongs to* $L_{q_0,\mathrm{loc}}(U, \mathbb{R}^k)$.

(iii) *There exist numbers* ε ($0 \le \varepsilon < \infty$) *and* t ($t_0 \le t < \infty$) *such that the deviation of* V *from the elliptic linear differential operators of* $\mathcal{O}_t$ *is at most* ε. *This means that for every* $\varepsilon' > \varepsilon$ *and every point* $x \in U$, *there are an open neighborhood* Z ($\subset U$) *of* x, *invertible linear mappings* $\beta : \mathbb{R}^m \to \mathbb{R}^m$ *and* $\omega : \mathbb{R}^n \to \mathbb{R}^n$, *and a measurable matrix-valued mapping* $\psi : U \to \mathbb{R}^{k \times k}$ *locally bounded in the sup-norm whose values* $\psi(x)$ *are invertible matrices for almost every* $x \in U$ *such that if, starting from a* $W^l_{q,\mathrm{loc}}$*-solution* $f : Z \to \mathbb{R}^m$ *to (7) in* Z, $q \ge 1$, *we construct the new mapping* $\bar{f} = \beta^{-1} \circ f \circ \omega^{-1}$ *then* $\bar{f}$ *is a* $W^l_{q,\mathrm{loc}}$*-solution in* $\omega(Z)$ *to the system*

$$\begin{gathered} \mathfrak{L}_{\psi,\beta,\omega}(z; v_{l-1}(\bar{f}, z); v^l(\bar{f}, z)) = \\ V_{\psi,\beta,\omega}(z; v_{l-1}(\bar{f}, z); v^l(\bar{f}, z)) + T_{\psi,\beta,\omega}(z; v_{l-1}(\bar{f}, z)) = \\ \psi(\omega^{-1}(z))V(\omega^{-1}(z); v_{l-1}(\beta \circ \bar{f} \circ \omega, \omega^{-1}(z)); v^l(\beta \circ \bar{f} \circ \omega, \omega^{-1}(z)))+ \\ \psi(\omega^{-1}(z))T(\omega^{-1}(z); v_{l-1}(\beta \circ \bar{f} \circ \omega, \omega^{-1}(z))) = 0 \end{gathered} \tag{8}$$

of the kind of (7); *moreover, there is an elliptic differential operator* $D = \sum\limits_{|p|=l} a_p\partial^p \in \mathcal{O}_t$ *meeting the condition*

$$|V_{\psi,\beta,\omega}(z; v_{l-1}; v^l) - \sum_{|p|=l} a_p\widetilde{v}^p| \le \varepsilon'|v^l|, \quad (v_{l-1}; v^l) \in \mathbb{R}^{N_l-n} \tag{9}$$

($\widetilde{v}^p$ is the vector in $\mathbb{R}^m$ with components $v_{p,\varkappa} = v_{p_l,\varkappa}$, $\varkappa = 1, 2, \dots, m$) *for almost every* $z \in \omega(Z)$.

Note that by (9)

$$|V_{\psi,\beta,\omega}(z; v_{l-1}(\check{f}, z); v^l(\check{f}, z)) - D\check{f}(z)| \le \varepsilon'|v^l(\check{f}, z)|$$

for almost every $z \in \omega(Z)$ if $\check{f} \in W^l_{1,\mathrm{loc}}(\omega(Z), \mathbb{R}^m)$.

Condition (iii) can be briefly characterized as follows: System (7) is locally reducible to form (8), (9) by Cordes type transformations (cf. [7]).

REMARK. *We further assume the mapping* $\mathfrak{L}$ *in (7) to be the best representative of mappings that coincide with* $\mathfrak{L}$ *almost everywhere in* U_1*:*

$$\mathfrak{L}_j(y) = \varlimsup_{r\searrow 0} \frac{1}{r^{N_l}v_{N_l}} \int\limits_{\{w\in\mathbb{R}^{N_l}, |w-y|<r\}} \mathfrak{L}_j(w)dw, \quad y \in U_1,$$

$j = 1, 2, \dots, k$ (v_{N_l} *is the volume of the* N_l*-dimensional unit ball* $B_{N_l}(0,1) = \{w \in \mathbb{R}^{N_l}, |w| < 1\}$*).*

Note that system (4) of linear partial differential equations, considered in Section 2, is an important particular case of systems (7). Indeed, we have

LEMMA 2. *If system* (4) *of linear differential equations with measurable coefficients and right-hand sides is uniformly elliptic, its higher coefficients satisfy the condition of slow variation, and the remaining coefficients and the right-hand sides are locally integrable to the power* $q_0 > n$ *(more exactly, the system meets conditions* ($\circ$) – ($\circ\circ\circ$) *of Section* 2 *with* $\varepsilon \ge 0$, $t \ge t_0$, *and* $q_0 > n$*), then the system satisfies* (i)–(iii) *where the parameters are* $\varepsilon\sqrt{kmn_l}$ ($n_l = \frac{(n+l-1)!}{l!(n-1)!}$), t, *and* q_0.

Thus, Beltrami systems, their various multidimensional generalizations as well as linear uniformly elliptic systems with continuous coefficients and right-hand sides are the systems of the kind of (7) satisfying (i)–(iii).

Furthermore, by Lemma 3 (see below), every general elliptic system (1) of nonlinear partial differential equations constructed from C^1-smooth functions $\mathfrak{L}_j$ can be treated locally as a system of the kind of (7) satisfying conditions (i)–(iii) with a small value of ε as regards the question of regularity for its C^l-smooth solutions.

LEMMA 3. *If* $f : U \to \mathbb{R}^m$, $U \Subset \mathbb{R}^n$, *is a* C^l*-solution to elliptic system* (1) *constructed from* C^1*-smooth functions* $\mathfrak{L}_j$, $j = 1, 2, \dots, k$, *then, for every point* $x_0 \in U$, *every two numbers* ε *and* q_0 *such that* $0 \le \varepsilon < \infty$ *and* $n < q_0 < \infty$, *and the number*

$$t = t(D) = t\Big(\sum_{|p|=l} a_p\partial^p\Big) = t\Big(\sum_{|p_l|=l} \Big\{\big(\partial_{v_{p_l,\varkappa}}\mathfrak{L}_j(y_0)\big)_{j=1,\dots,k\,\varkappa=1,\dots,m}\Big\}\partial^{p_l}\Big), \tag{10}$$

$$y_0 = (x_0; \dots, \partial^{p_0} f_\varkappa(x_0), \dots; \dots, \partial^{p_1} f_\varkappa(x_0), \dots; \dots; \dots, \partial^{p_l} f_\varkappa(x_0), \dots),$$

being the least of λ *such that*

$$|a_p^{j\varkappa}| \le \lambda, \quad j = 1, 2, \dots, k, \quad \varkappa = 1, 2, \dots, m, \quad |p| = l,$$
$$\inf_{\zeta \in \mathbb{R}^n, u \in \mathbb{R}^m, |\zeta|=1, |u|=1} \Big| \sum_{|p|=l} \zeta^p a_p u \Big| \ge 1/\lambda$$

($a_p^{j\varkappa}$ *are the coefficients of the operator* D *in* (10)*), there exists a neighborhood* U_{x_0} $(\subset U)$ *of* x_0 *for which we have the following: The restriction* $(f - P^l_{f,x_0})|_{U_{x_0}}$ *of the difference between* f *and its Taylor polynomial* P^l_{f,x_0} *of degree* l *at* x_0 *to* U_{x_0} *is a solution to a system of the kind of* (7) *locally close to elliptic systems of linear partial differential equations with constant coefficients; moreover, the proximity is described by conditions* (i)–(iii) *with the indicated values of* ε, q_0, *and* t.

Note that the proof of Theorem 1 on a Hölder continuity of higher derivatives of elliptic C^l-smooth solutions to systems (1) of nonlinear partial differential equations constructed from C^1-smooth functions $\mathfrak{L}_j$ is based on Lemma 3 and the main result of this section (and of the whole article), Theorem 9, which will be discussed below.

If the system is linear, i.e., has the form (4), then Lemma 3 can be strengthened as follows:

LEMMA 4. *Suppose that the coefficients* $a_p^{j\varkappa}$ *and right-hand sides* h_j *of system* (4), $j = 1, 2, \dots, k$, $\varkappa = 1, 2, \dots, m$, $|p| \le l$, *are continuous functions and the system itself is elliptic (this means the same as in Theorem* 7*). Then, for every bounded open set* $U_0 \subset \mathbb{R}^n$ *lying in* U *together with its closure* $\operatorname{cl} U_0$, *there exists a number* $t = t_{U_0} \ge t_0$ *meeting the following: if* $n < q_0 < \infty$ *and* $0 \le \varepsilon < \infty$ *then system* (4) *satisfies* $(\circ) - (\circ\circ\circ)$ *in* $U_0 \times \prod_{\nu=0}^{l} (\mathbb{R}^m)^{n_\nu}$ *with the indicated values of* ε, q_0, *and* t.

REMARK. *Lemma 4 extends (with corresponding alterations) also to the case of elliptic systems (4), the leading coefficients* $a_p^{j\varkappa}$, $|p| = l$, $j = 1, 2, \dots, k$, $\varkappa = 1, 2, \dots, m$, *of which are continuous, and the remaining coefficients and the right-hand sides are integrable to the power* $q_0 > n$ *locally in* U *(in this case, ellipticity of system (4) means the same as in Theorems 7 and 8, and Lemma 4, i.e.,* $\operatorname{rank}\big\{\sum_{|p|=l} \zeta^p a_p(x)\big\} = m$ *for all* $\zeta \in \mathbb{R}^n \setminus \{0\}$ *and* $x \in U$*). A system of this kind has the following property: Let* U_0 *be an open bounded set in* $\mathbb{R}^n$ *such that* $\operatorname{cl} U_0 \subset \mathbb{R}^n$. *There exists a number* $t = t_{U_0} \ge t_0$ *such that if* $0 \le \varepsilon < \infty$ *then the system satisfies conditions* $(\circ) - (\circ\circ\circ)$ *in* $U_0 \times \prod_{\nu=0}^{l} (\mathbb{R}^m)^{n_\nu}$ *with these values of* ε *and* t *and with the above-indicated value of* q_0.

The main assertion of this section is

THEOREM 9.*Suppose that* $1 < r < \infty$. *There exists a function* $\Gamma_r = \Gamma_r^{n,m,k,l} : [t_0, \infty[\to]0, \infty[$ *satisfying the following: if* ε (≥ 0), q_0 $(> n)$ *and* t $(\ge t_0)$ *are such that* $\varepsilon < \frac{\Gamma_r(t)}{q_0}$, *and system* (7) *meets conditions* (i)–(iii) *with these* ε, q_0, *and* t, *then every* $W^l_{r,\mathrm{loc}}$*-solution to the system is its* $W^l_{q_0,\mathrm{loc}}$*-solution.*

Theorem 9 and the embedding theorems for the Sobolev spaces imply (cf. Theorem 5)

COROLLARY. *If the conditions of Theorem 9 are fulfilled then each* $W_{r,\mathrm{loc}}^l$*-solution to system* (7) *belongs to* $C_{\mathrm{loc}}^{(l-1)+(1-n/q_0)}$.

In the case when the order l of system (7) is equal to 1 Theorem 9 can be given a stronger form:

THEOREM 10. *If* $l = 1$, $r > 1$ *and the conditions of Theorem* 9 *are fulfilled with* β *and* ω *quasi-isometries in* (iii) *then the theorem remains true.*

In conclusion, note that by Lemma 2 on the relation between systems (4) of linear partial differential equations and systems of the kind of (7), Theorem 2 on the W_q^l-regularity of solutions to system (4) satisfying conditions $(\circ) - (\circ\circ\circ)$ ensues directly from Theorem 9. Furthermore, Theorem 9 makes it possible to take $\mathcal{C}_r(t) = \frac{\Gamma_r(t)}{\sqrt{kmn_l}}\frac{1}{q_0}$ in Theorem 3, that is, the quantity $\mathcal{E}_{t,r}(q_0)$ has the following lower bound:

$$\frac{\Gamma_r(t)}{\sqrt{kmn_l}}\frac{1}{q_0} \le \mathcal{E}_{t,r}(q_0).$$

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